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Green's functions, sum rules and matrix elements for SUSY partners

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Abstract
It is now well known that the solutions in the Schrödinger equations for two potentials which are supersymmetric partners are linked by intertwining relations involving first-order differential operators. In this paper, we explore the consequence of this linkage for the Green's functions corresponding to SUSY partners and the relation between the sums over the inverses of the eigenvalues for the two potentials. We also establish some relations between the matrix elements of certain operators evaluated between the eigenstates of two partner potentials. We show that there can be circumstances where some matrix elements vanish as a consequence of the presence of supersymmetry.

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1. Introduction

Supersymmetric quantum mechanics (SUSYQM) is the study of the property of Hamiltonians linked by the algebra of supersymmetry. This possibility was first suggested in a paper by Witten (1981). It has been shown (Andrianov *et al* 1984, Sukumar 1985a, 1985b) that the simplest non-trivial realization of the algebra of supersymmetry leads to the result that a one-dimensional non-relativistic Hamiltonian can have a partner $\tilde{H}$ whose spectrum is either identical to that of H or misses the ground state of H or has an additional boundstate below that of H. In the past 15 years, SUSYQM has been used to study a variety of physical systems from electrons in magnetic fields (Khare and Maharana 1984) to the Dirac equation (Sukumar 1985c) for the hydrogen atom and to explaining the relation between deep and shallow potentials used in the study of problems in nuclear physics (Baye 1987, 1994). The variety of applications in atomic, nuclear and solidstate physics is now very large.

For one-dimensional Schrödinger equations, it is well known that it is possible to construct a Green's function satisfying homogeneous boundary conditions by solving an inhomogeneous differential equation with a delta function source (Morse and Feshbach 1953, Sukumar 1990).

There are two possible representations of the Green's function either in terms of solutions of a homogeneous differential equation at an energy which is not an eigenenergy or in terms of a complete set of eigenfunctions of the homogeneous differential equation. The existence of two representations of G leads to certain integral equalities relating an integral of the Green's function to a sum involving spectral parameters. Since SUSYQM establishes a link between the solutions of two Schrödinger equations connected by the SUSY algebra, this implies a relation between the corresponding Green's functions. In section 2 of this paper, we study this relationship and derive an important relation between the spectral densities of two systems linked by SUSY algebra.

The intertwining relation between the eigenstates of SUSY partner potentials also has the consequence that the matrix elements of operators for the two SUSY components are also linked. In section 3 of the paper, we study the relation between the two sets of matrix elements. We show that the presence of supersymmetry may lead to the vanishing of certain matrix elements for one of the two SUSY partner potentials. Two exactly solvable problems which exhibit this property are discussed in section 4 of the paper. Section 5 contains the conclusions. Units in which $\hbar = 1$ and the mass $\mu = \frac{1}{2}$ are used throughout the paper so that $\frac{\hbar^2}{2\mu} = 1$.

2. Green's functions for SUSY partners

SUSYQM has established that for every one-dimensional Hamiltonian H given by

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \tag{1}$$

with eigenstates Ψ_n with eigenvalue E_n there is a partner Hamiltonian $\tilde{H}$

$$\tilde{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tilde{V} \tag{2}$$

such that their spectral properties are related. The two Hamiltonians defined by

$$H = A^+(E_0)A^-(E_0) + E_0 \qquad \tilde{H} = A^-(E_0)A^+(E_0) + E_0 \tag{3}$$

where

$$A^\pm = \left(\pm\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{\Psi_0}(x)\frac{\mathrm{d}}{\mathrm{d}x}\Psi_0(x)\right) \tag{4}$$

have exactly the same spectrum except that E_0 is not an eigenvalue for $\tilde{H}$. The two potentials are related by

$$\tilde{V} = V - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\Psi_0 \tag{5}$$

and the normalized eigenstates of the two hamiltonians are related by

$$\tilde{\Psi}_n = (E_n - E_0)^{-\frac{1}{2}}A^-\Psi_n \qquad \Psi_n = (E_n - E_0)^{-\frac{1}{2}}A^+\tilde{\Psi}_n. \tag{6}$$

These intertwining relationships are valid for all boundstates except the ground state of H corresponding to eigenvalue E_0 and may be extended to all states including scattering states and solutions corresponding to energies which are not eigenvalues.

Green's functions G and $\tilde{G}$ may be associated with the Schrödinger equations corresponding to the Hamiltonians H and $\tilde{H}$ which are solutions to

$$\left(-\frac{\partial^2}{\partial x^2} + V - E\right)G(x,y) = -\delta(x-y) \tag{7}$$

$$\left(-\frac{\partial^2}{\partial x^2}+\tilde{V}-E\right)\tilde{G}(x,y)=-\delta(x-y) \tag{8}$$

where E can take any value except one of the eigenvalues E_n and in this paper we consider values of $E < E_0$. G may be constructed from two solutions of the homogeneous differential equation

$$\left(-\frac{\partial^2}{\partial x^2}+V-E\right)\Phi=0 \tag{9}$$

each of which satisfies one of the boundary conditions satisfied by the eigenstates Ψ_n. Let $x_0 \leqslant x \leqslant x_1$ be the domain. Let Φ_1 satisfy the same boundary condition as Ψ_n at x_0 and Φ_2 satisfy the same boundary condition as Ψ_n at x_1. Let

$$\Phi_1(x_0)=0 \qquad \Phi_2=\Phi_1\int_{x_1}^{x}\frac{\mathrm{d}z}{\Phi_1^2} \qquad \Phi_2(x_1)=0 \tag{10}$$

so that the Wronskian of Φ_1 and Φ_2 equals 1. In terms of such solutions G may be constructed in the form

$$G(x,y)=\Phi_1(x)\Phi_2(y)\theta(y-x)+\Phi_1(y)\Phi_2(x)\theta(x-y)=\Phi_1(x_<)\Phi_2(x_>) \tag{11}$$

where $x_<(x_>)$ is the smaller (larger) of (x, y) and θ is the unit step function with the property that $\theta(z)$ has value 0 if $z \leqslant 0$ and has value 1 if $z \geqslant 0$. Using the property that the derivative of a step function is a delta function, it is easy to show that G defined as above satisfies the inhomogeneous differential equation (equation (7)) for $G(x, y)$. An alternative representation of G in terms of a complete set of normalized eigenstates of H is

$$G(x,y)=\sum_{n=0}^{\infty}\frac{\Psi_n(x)\Psi_n(y)}{E-E_n}+\int_0^{\infty}\frac{\Psi_k(x)\Psi_k(y)}{E-k^2}\rho(k)\,\mathrm{d}k \tag{12}$$

where $\rho(k)$ is the spectral density for scattering states. The above equation includes an integral over continuum states even though the continuum states do not satisfy boundstate boundary conditions in the asymptotic region. It is well established in the theory of Fourier analysis that in order to fulfil boundary conditions, it is not necessary that each individual term satisfies the boundary conditions but that the sum or the integral satisfy the required boundary conditions. Using the completeness relation

$$\sum_{n=0}^{\infty}\Psi_n(x)\Psi_n(y)+\int_0^{\infty}\Psi_k(x)\Psi_k(y)\rho(k)\,\mathrm{d}k=\delta(x-y) \tag{13}$$

it is easy to show that G represented in this form also satisfies equation (7) which provides a justification for the inclusion of the integral over continuum states in equation (12). We adopt the convention that the continuum states can be ortho-normalized over a unit interval . The existence of two representations of G leads to the integral relation

$$\int_{x_0}^{x_1}G(x,x)\,\mathrm{d}x=\int_{x_0}^{x_1}\Phi_1(x)\Phi_2(x)\,\mathrm{d}x=\sum_{n=0}^{\infty}\frac{1}{E-E_n}+L\int_0^{\infty}\frac{\rho(k)}{E-k^2}\,\mathrm{d}k \tag{14}$$

when the orthonormality of the eigenstates is used. In equation (14), the parameter L corresponds to the integration interval $x_1 - x_0$. The corresponding relation for the SUSY partner is then given by

$$\int_{x_0}^{x_1}\tilde{G}(x,x)\,\mathrm{d}x=\int_{x_0}^{x_1}\tilde{\Phi}_1(x)\tilde{\Phi}_2(x)\,\mathrm{d}x=\sum_{n=1}^{\infty}\frac{1}{E-E_n}+L\int_0^{\infty}\frac{\tilde{\rho}(k)}{E-k^2}\,\mathrm{d}k \tag{15}$$

in which the sum now starts at $n = 1$ and $\tilde{\rho}$ is the spectral density for the potential $\tilde{V}$. For $E < E_0$, the intertwining relation between the solutions in the two potentials may be written in the form

$$\tilde{\Phi}_{1,2} = -b\Psi_0 \frac{\mathrm{d}}{\mathrm{d}x}\frac{\Phi_{1,2}}{\Psi_0} \qquad \Phi_{1,2} = b\frac{1}{\Psi_0}\frac{\mathrm{d}}{\mathrm{d}x}(\tilde{\Phi}_{1,2}\Psi_0) \qquad b = \mathrm{i}(E_0 - E)^{-\frac{1}{2}}. \tag{16}$$

These relations imply that

$$\Phi_1\tilde{\Phi}_2 - \Phi_2\tilde{\Phi}_1 = -b\left(\Phi_1\frac{\mathrm{d}}{\mathrm{d}x}\Phi_2 - \Phi_2\frac{\mathrm{d}}{\mathrm{d}x}\Phi_1\right) = -b. \tag{17}$$

We now consider the function F defined by

$$F = \frac{1}{b}(\Phi_1\tilde{\Phi}_2 + \Phi_2\tilde{\Phi}_1) \tag{18}$$

in terms of which

$$\Phi_1\tilde{\Phi}_2 = \frac{b}{2}(F - 1) \qquad \Phi_2\tilde{\Phi}_1 = \frac{b}{2}(F + 1) \tag{19}$$

and

$$G\tilde{G} = \Phi_1\Phi_2\tilde{\Phi}_1\tilde{\Phi}_2 = \frac{b^2}{4}(F^2 - 1). \tag{20}$$

These equations show that

$$\frac{\tilde{\Phi}_2}{\Phi_2} = \frac{b}{2}\frac{F - 1}{G} \qquad \frac{\tilde{\Phi}_1}{\Phi_1} = \frac{b}{2}\frac{F + 1}{G}. \tag{21}$$

It can be shown using equations (11) and (16) that

$$\Psi_0^2\frac{\mathrm{d}}{\mathrm{d}x}\frac{G(x, x)}{\Psi_0^2} = -\frac{1}{b}(\Phi_1\tilde{\Phi}_2 + \Phi_2\tilde{\Phi}_1) = -F. \tag{22}$$

Furthermore

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}x} &= \frac{1}{b}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\Phi_1}{\Psi_0}\tilde{\Phi}_2\Psi_0 + \frac{\Phi_2}{\Psi_0}\tilde{\Phi}_1\Psi_0\right) \\ &= \frac{2}{b^2}(-\tilde{\Phi}_1\tilde{\Phi}_2 + \Phi_1\Phi_2) = \frac{2}{b^2}(-\tilde{G} + G)\end{aligned} \tag{23}$$

which is an equation that can be used to study the relation between equations (14) and (15). The boundary conditions at $x = x_1$ are

$$Lt_{x\to x_1}G \to 0 \quad \tilde{G} \to 0 \quad \Phi_2 \to 0 \quad \tilde{\Phi}_2 \to 0 \quad \Phi_1 \neq 0 \quad \tilde{\Phi}_1 \neq 0 \tag{24}$$

which when taken together with equations (20) and (21) show that

$$Lt_{x\to x_1}F^2 - 1 = 0 \qquad F(x_1) = +1. \tag{25}$$

Similarly by examining the boundary conditions at $x = x_0$

$$Lt_{x\to x_0}G \to 0 \quad \tilde{G} \to 0 \quad \Phi_1 \to 0 \quad \tilde{\Phi}_1 \to 0 \quad \Phi_2 \neq 0 \quad \tilde{\Phi}_2 \neq 0 \tag{26}$$

and taking into account the conditions in equations (20) and (21), we can infer that

$$Lt_{x\to x_0}F^2 - 1 = 0 \qquad F(x_0) = -1. \tag{27}$$

The boundary values of F given in equations (25) and (27) when taken together with the differential equation for F in equation (23) give the relation

$$\int_{x_0}^{x_1}(G - \tilde{G})\,\mathrm{d}x = b^2 = \frac{1}{E - E_0}. \tag{28}$$

By comparing this with the relations in equations (14) and (15), we arrive at the relation

$$L\int_0^\infty \frac{\rho(k)-\tilde{\rho}(k)}{E-k^2}\,\mathrm{d}k = 0 \tag{29}$$

from which we can conclude that since the denominator is negative definite when $E < E_0$ and the spectral densities are positive semi-definite we must have

$$\rho(k) = \tilde{\rho}(k) \tag{30}$$

which shows that for scattering energies, the spectral densities for the two SUSY partners are identical. For confining potentials with no scattering states ρ and $\tilde{\rho}$ both vanish and for this case the trace relation between the Greens functions in equation (28) is in accord with equations (14) and (15) when the $n = 0$ term missing in the second sum is taken into account. Thus, we have established a relation between the integrals of the equiposition Green's functions and sums involving the energy eigenvalues and the spectral densities for the two SUSY partner potentials.

3. Matrix elements for SUSY partners

The intertwining relations between the eigenfunctions for the two SUSY partners may be given in a differential form as in equation (6) or by using the Wronskian relation between two eigensolutions and the differential equation satisfied by the eigenfunctions can be given in an integral form as

$$\begin{aligned}\tilde{\Psi}_j &= -\epsilon_{j0}^{-\frac{1}{2}}\Psi_0\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\Psi_j}{\Psi_0}\right) = \epsilon_{j0}^{\frac{1}{2}}\frac{1}{\Psi_0}\int^x \Psi_0\Psi_j\,\mathrm{d}y \\ \Psi_j &= \epsilon_{j0}^{-\frac{1}{2}}\frac{1}{\Psi_0}\frac{\mathrm{d}}{\mathrm{d}x}(\tilde{\Psi}_j\Psi_0) = -\epsilon_{j0}^{\frac{1}{2}}\Psi_0\int^x \frac{\tilde{\Psi}_j}{\Psi_0}\,\mathrm{d}y\end{aligned} \tag{31}$$

where $\epsilon_{j0} = (E_j - E_0)$. We now consider the matrix elements of an operator A which is a function of the position x but not of the momentum, i.e $[A, x] = 0$. For such an operator, the matrix element taken between the eigenstates of $\tilde{H}$ may be expressed in terms of the eigenstates of H using the above relations and integration by parts in the form

$$\begin{aligned}\tilde{A}_{jk} &= \int_{x_0}^{x_1} \tilde{\Psi}_j A\tilde{\Psi}_k\,\mathrm{d}x \\ &= -\epsilon_{j0}^{-\frac{1}{2}}\epsilon_{k0}^{-\frac{1}{2}}\int_{x_0}^{x_1}\frac{\Psi_j}{\Psi_0}\frac{\mathrm{d}}{\mathrm{d}x}\left(A\left(\Psi_0\frac{\mathrm{d}}{\mathrm{d}x}\Psi_k - \Psi_k\frac{\mathrm{d}}{\mathrm{d}x}\Psi_0\right)\right)\mathrm{d}x\end{aligned} \tag{32}$$

which can be further simplified using the Schrödinger equation satisfied by the eigenstates Ψ_k to the form

$$\tilde{A}_{jk} = \left(\frac{\epsilon_{k0}}{\epsilon_{j0}}\right)^{\frac{1}{2}} A_{jk} - \epsilon_{j0}^{-\frac{1}{2}}\epsilon_{k0}^{-\frac{1}{2}}\int_{x_0}^{x_1}\frac{\mathrm{d}A}{\mathrm{d}x}\Psi_0\Psi_j\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\Psi_k}{\Psi_0}\right)\mathrm{d}x. \tag{33}$$

The Hermiticity of A may be used to give an alternate expression for the matrix element by swapping the indices j and k. By taking a symmetric average of the two expressions it can be shown that

$$\tilde{A}_{jk} = \frac{1}{2}\epsilon_{j0}^{-\frac{1}{2}}\epsilon_{k0}^{-\frac{1}{2}}\int_{x_0}^{x_1}\Psi_j\Psi_k\left(\epsilon_{j0}+\epsilon_{k0}+\frac{\mathrm{d}^2}{\mathrm{d}x^2}+2\frac{1}{\Psi_0}\frac{\mathrm{d}\Psi_0}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\right)A\,\mathrm{d}x. \tag{34}$$

This can be further simplified using

$$\frac{1}{\Psi_0}[A, H]\Psi_0 = \frac{1}{\Psi_0}(E_0 - H)(A\Psi_0) = \frac{\mathrm{d}^2 A}{\mathrm{d}x^2} + 2\frac{\mathrm{d}A}{\mathrm{d}x}\frac{1}{\Psi_0}\frac{\mathrm{d}}{\mathrm{d}x}\Psi_0 \tag{35}$$

which leads to the equivalent expressions

$$\tilde{A}_{jk} = \frac{1}{2}\epsilon_{j0}^{-\frac{1}{2}}\epsilon_{k0}^{-\frac{1}{2}} \int_{x_0}^{x_1} \frac{\Psi_j \Psi_k}{\Psi_0}(E_j + E_k - E_0 - H)(A\Psi_0)\,\mathrm{d}x \tag{36}$$

$$\tilde{A}_{jk} = \frac{1}{2}\epsilon_{j0}^{-\frac{1}{2}}\epsilon_{k0}^{-\frac{1}{2}} \left((\epsilon_{j0} + \epsilon_{k0})A_{jk} + \int_{x_0}^{x_1} \frac{\Psi_j \Psi_k}{\Psi_0}[A, H]\Psi_0\,\mathrm{d}x\right). \tag{37}$$

It is also clear from equation (36) that for the special case when $A = \Psi_n/\Psi_0$ there is a simple relation between the matrix elements of the two SUSY partners which is given by

$$\int_{x_0}^{x_1} \tilde{\Psi}_j \frac{\Psi_n}{\Psi_0}\tilde{\Psi}_k\,\mathrm{d}x = \frac{E_j + E_k - E_0 - E_n}{2\epsilon_{j0}^{\frac{1}{2}}\epsilon_{k0}^{\frac{1}{2}}} \int_{x_0}^{x_1} \Psi_j \frac{\Psi_n}{\Psi_0}\Psi_k\,\mathrm{d}x. \tag{38}$$

The above-derived expression shows that there can be circumstances where the alignment of energy levels is such that the factor in front of the integral on the right-hand side of the equation vanishes. This will lead to the possibility that the matrix element for the SUSY partner vanishes. The vanishing of a matrix element is usually associated with the presence of some symmetry in the system under consideration or because of some accidental circumstance. In the case under consideration, the vanishing of the matrix element is a real consequence of the supersymmetric connection between two potentials. Two examples of such a vanishing of the matrix elements will be discussed in the next section.

If we consider an operator B which is a function of momentum but not a function of position, i.e $[B, p] = 0$, then by using a procedure analogous to that for A it can be established that

$$\tilde{B}_{jk} = \left(\frac{\epsilon_{k0}}{\epsilon_{j0}}\right)^{\frac{1}{2}} B_{jk} + \epsilon_{j0}^{-\frac{1}{2}}\epsilon_{k0}^{-\frac{1}{2}} \int_{x_0}^{x_1} \frac{\Psi_j}{\Psi_0}\left(\Psi_0 B\frac{\mathrm{d}\Psi_0}{\mathrm{d}x} - \frac{\mathrm{d}\Psi_0}{\mathrm{d}x}B\Psi_0\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\Psi_k}{\Psi_0}\right). \tag{39}$$

If the operator has definite Hermitian or anti-Hermitian character, then by swapping the indices j and k it is possible to find alternative expressions which exploit the relationship between $\tilde{B}_{jk}$ and $\tilde{B}_{kj}$.

So far we have discussed the matrix elements of operators which are either pure functions of x such as A which commute with other functions of x or operators such as B which have only derivatives present in them. It is possible to extend the discussion to more general operators using the methods outlined in this section.

4. Examples of vanishing matrix elements

One of the main results of the previous section, namely equation (38), shows that when the condition $E_j + E_k = E_n + E_0$ is fulfilled then the matrix element of the function Ψ_n/Ψ_0 taken between the eigenstates j and k of $\tilde{H}$ vanishes. This condition for the energies can happen accidentally for any H for some particular choice of indices j, k and n. In this section, we show two examples where this condition will be met systematically.

4.1. Simple harmonic oscillator

For $V = \omega^2 x^2/4$, the eigenvalues are $E_n = (n + 1/2)\omega$ and the ground state wavefunction is $\Psi_0 \sim \exp(-\omega x^2/4)$. The SUSY partner potential from equation (5) is $\tilde{V} = V + \omega$ which is the oscillator shifted vertically by ω. Since the change in potential is exactly equal to the spacing of the energy levels in this example, the relation between the eigenstates is $\tilde{\Psi}_{n+1} = \Psi_n$. If we choose $A = V$ then using the property that the eigenstates are products of Gaussians and Hermite polynomials A may be expressed as a linear combination of the eigenstates with quantum numbers 0 and 2 and hence

$$A\Psi_0 = V\Psi_0 = V_{20}\Psi_2 + V_{00}\Psi_0. \tag{40}$$

The expansion given above leads to the result that

$$(E_0 - H)A\Psi_0 = V_{20}(E_0 - E_2)\Psi_2 = -2\omega(V - V_{00})\Psi_0. \tag{41}$$

By choosing $j = k = m + 1$ equation (35) can then be used to establish the relation

$$\tilde{V}_{m+1,m+1} = V_{mm} = V_{m+1,m+1} - \frac{1}{m+1}(V_{m+1,m+1} - V_{00}) \tag{42}$$

which leads to the expression

$$V_{mm} = \frac{m}{m+1}V_{m+1,m+1} + \frac{1}{m+1}V_{00}. \tag{43}$$

This equation is clearly satisfied by the well-known oscillator matrix elements

$$V_{mm} = \left(m + \frac{1}{2}\right)\frac{\omega}{2}. \tag{44}$$

We have shown that since the oscillator has the property that its SUSY partner is also an oscillator which is just shifted in energy, the relation between the matrix elements of SUSY partners simply becomes a condition on the matrix elements for the oscillator. If we now consider the case $A = \Psi_n/\Psi_0$ then equation (38) becomes

$$\tilde{A}_{jk} = A_{j-1,k-1} = \frac{j + k - n}{2\sqrt{j}\sqrt{k}}A_{jk} \tag{45}$$

and the matrix element vanishes if $j + k = n$. A particularly striking example of this arises if we set $j = k = m + 1, n = 2m + 2$ which leads to the condition that the harmonic oscillator wavefunctions must satisfy

$$\int_{-\infty}^{\infty} \Psi_m^2 \frac{\Psi_{2m+2}}{\Psi_0}\,\mathrm{d}x = 0. \tag{46}$$

This result is in agreement with a general result involving the products of three Hermite polynomials (Gradshteyn and Ryzhik 1965) in the form

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} H_k H_m H_n\,\mathrm{d}x &= 2^s\sqrt{\pi}\frac{\Gamma(k+1)}{\Gamma(s+1-k)}\frac{\Gamma(m+1)}{\Gamma(s+1-m)}\frac{\Gamma(n+1)}{\Gamma(s+1-n)} \\ s &= m + n + k = \text{even}.\end{aligned} \tag{47}$$

The vanishing of the integral given in equation (46) cannot be explained from any obvious spatial symmetry of the wavefunctions, but is a consequence of an underlying supersymmetry.

4.2. Particle in a box

For a particle confined inside a box with an infinite potential wall at $|x| = \pi/2$, the eigenvalues are $E_n = (n + 1)^2$. The ground state wavefunction is $\Psi_0 \sim \cos x$. The SUSY partner is $\tilde{V} = 2\sec^2 x$. By eliminating one state at a time, it is possible to produce a sequence of potentials $V_m = m(m + 1)\sec^2 x$ such that two potentials with adjacent values of m are SUSY partners. The lowest member of this $\sec^2$ sequence is $V_1 = \tilde{V}$. The eigenstates of any member of this sequence may be related to the eigenstates of the particle in a box through a chain of intertwining relations arising from equation (6). The condition for the vanishing of a matrix element arising from equation (38) in this example becomes

$$(j + 1)^2 + (k + 1)^2 = 1 + (n + 1)^2. \tag{48}$$

There are infinitely many sets of values of (j, k, n) for which this condition can be met. Some examples are (4, 4, 6), (3, 6, 7), (7, 8, 11) and (10, 12, 16). Each of these sets of values will lead to the vanishing of an integral. For example, the set $j = 4, k = 4, n = 6$ corresponds to the eigenfunctions

$$\tilde{\Psi}_4 \sim -\cos x \frac{\mathrm{d}}{\mathrm{d}x} \frac{\cos 5x}{\cos x} \qquad \Psi_6 \sim \cos 7x. \tag{49}$$

Use of these eigenfunctions in equation (36) leads to

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (5 \sin 5x - \cos 5x \tan x)^2 \frac{\cos 7x}{\cos x} \, \mathrm{d}x = 0. \tag{50}$$

The vanishing of this integral can be explicitly verified by a long calculation. Again the vanishing of this integral cannot be explained by an easily identifiable geometrical symmetry. The vanishing of this integral is a consequence of a supersymmetric link between the eigenfunctions. The argument used here shows that there are many sets of values of (j, k, n) for which some matrix elements vanish and this feature has its dynamical origin in the presence of an underlying supersymmetry.

5. Conclusions

In this paper, it has been shown that the intertwining relationship between the eigenfunctions of two SUSY partner Hamiltonians has a number of significant consequences. It has been shown that the Green's functions for the two systems are related. The trace formulae for the Green's functions which link the spectral parameters for the SUSY partners have been constructed and it has been proved that the spectral density for scattering states is the same. For the case of the confining potentials which do not have any scattering states, the trace formulae are in accord with the sums over the inverses of the eigenvalues when the missing of the ground state of one of the SUSY partners is taken into account.

It has been shown that the intertwining relationships also imply that there are circumstances when the matrix elements of some operators taken between the eigenstates of one of the SUSY partners will vanish while the corresponding matrix element for the other partner Hamiltonian is non-vanishing. This feature is a consequence of the underlying supersymmetry linking the two Hamiltonians. In the case of the simple harmonic oscillator, certain integrals involving the eigenfunctions are shown to vanish identically. This is associated with the feature that the supersymmetric partner to the oscillator is also the oscillator but only shifted in energy. In the case of a free particle confined between infinite walls, it has been shown that some matrix elements vanish when the energy levels involved satisfy a simple condition. These are two examples of exactly solvable systems in which the vanishing of the matrix element can be exhibited explicitly. However, the results derived in this paper are quite general and show that there are still many surprising features lurking in supersymmetric quantum mechanics.

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